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EXISTENCE OF ASYMPTOTICALLY ALMOST AUTOMORPHIC SOLUTIONS FOR A THIRD ORDER DIFFERENTIAL EQUATION

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Abstract. This paper deals with results on existence of asymptotically almost automorphic solutions for a third order in time abstract differential equation which model, on one side, high intensity ultrasound in acoustic wave propagation, while on the other side, vibrations of flexible structures possessing internal material damping. We established the asymptotically almost automorphy of the output solution subject to the asymptotically almost automorphy of the input disturbance.

1. Introduction

It is well known, that the dynamics of linear vibrations of elastic structures are mathematically governed by the wave equation. However, the dynamics of elastic vibrations of flexible structures are actually nonlinear in practice. In 1998, Bose and Gorain [6] studied a more realistic model of vibrations of elastic structure in which the stress is not simply proportional to the strain. As a result, they shown that the dynamics of vibrations of elastic structures are governed by the following third order differential equation

$$\alpha u'''(t) + u''(t) - \beta\Delta u(t) - \gamma\Delta u'(t) = 0, \qquad t \geq 0 \tag{1.1}$$

with suitable boundary and initial conditions, and where α, β, γ are positive constants. This third order in time equation displays, even in the linear version, a variety of dynamical behaviors for their solutions that depend on the physical parameters in the equation. These range from non-existence and instability to exponential stability (in time)[21]. Concerning qualitative properties, Bose and Gorain studied boundary stabilization and obtained the explicit exponential energy decay rate for the solution of (1.1) subject to mixed boundary conditions (see [6, 7, 16, 17] and references therein). Motivated by these works, abstract linear equations of the form

$$\alpha u'''(t) + u''(t) - \beta A u(t) - \gamma A u'(t) = f(t), \quad \alpha, \beta, \gamma \in \mathbb{R}_+, \quad t \geq 0, \tag{1.2}$$

where A is a closed linear operator acting in a Banach space X and f is a X-valued function has been treated in recent papers [9, 14, 15, 11]. We emphasize that

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the abstract Cauchy problem associated with (1.2) is in general ill-posed, see e.g. [30]. We mention that models related to (1.2) have been recently also considered in [21], where (1.1) is called Moore-Gibson-Thompson equation, and the nonlinear version is referred to as the Jordan-Moore-Gibson-Thompson-Westervelt equation. In [21] equation (1.1) arise as a model in acoustics, more precisely in high intensity ultrasound. The results in [21] for (1.2) assumed that A is a selfadjoint operator defined on a Hilbert space H and rewrite the equation as a first order abstract system on the phase space $D(A^{1/2}) \times D(A^{1/2}) \times H$.

However it is well known that in order to analyze higher order equations in an abstract setting, a direct approach leads in some situations to better results than those obtained by a reduction to a first-order equation, see e.g. [8] and [13].

Our purpose in this paper is analyze and to prove the existence of asymptotically almost automorphic mild solutions for an abstract semilinear equation of the form

$$\alpha u'''(t) + u''(t) - \beta Au(t) - \gamma Au'(t) = f(t, u(t), u'(t), u''(t)), \quad \alpha, \beta, \gamma \in \mathbb{R}_+, \tag{1.3}$$

with appropriate initial conditions. The motivation for incorporating f as an input disturbance in the governing differential equation arises from the fact that very small amount of these, are always present in real materials as long as the system vibrates. Hence, is also reasonable the study of existence of asymptotically almost automorphic solutions when $f(t, x)$ is asymptotically almost automorphic in t; that is, asymptotically almost automorphic stability of the system.

A surprising fact is that in order to get asymptotic behavior, some initial conditions should be forced to be zero. This leads to an unexpected property that is not present in the study of the same qualitative property for the Cauchy problem of order less than 3, see [1].

To achieve our goal we use a mixed method, combining tools of certain strongly continuous families in operator theory, introduced in [11], and fixed point theory.

This paper is organized as follows: The preliminary Section 2 collects results essentially contained in [23] and standard literature of almost automorphic and asymptotically almost automorhic functions (see [18], [19]). In particular we establish a result of composition for asymptotically almost automorphic functions (see Lemma 2.7) which is very important in our investigations. In Section 3 we first recall from [11] sufficient conditions for existence of solutions for equation (1.3). In fact, Proposition 3.1 gives a complete description of the solutions in terms of (α, β, γ)-regularized families. It corresponds to an extension of the standard variation of parameters formula. Then, we study conditions for existence and uniqueness of asymptotically almost automorphic solutions. We have two situations: In the linear case, we can ensure conditions for existence of asymptotically almost automorphic solution (see Theorem 3.3). For the semilinear case, we establish sufficient conditions for existence of asymptotically almost automorphic mild solutions (see

Theorem 3.3, Theorem 3.5, Theorem 3.9 and Theorem 3.11). In an special case, we are also able to prove existence of mild solution with nonlocal conditions (Theorem 4.12). Finally, we show that our abstract results apply to equation (1.3) in case of $A = \Delta$, the Laplacian.

2. Preliminaries

Let $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha \neq 0$ be given. In what follows we denote

$$k(t) = \frac{1}{\alpha}\int_0^t (t-s)e^{-s/\alpha}ds = -\alpha + t + \alpha e^{-t/\alpha}, \quad t \in \mathbb{R}_+$$

and

$$a(t) = \beta k(t) + \frac{\gamma}{\alpha}\int_0^t e^{-s/\alpha}ds = -(\alpha\beta - \gamma) + \beta t + (\alpha\beta - \gamma)e^{-t/\alpha}, \quad t \in \mathbb{R}_+ .$$

In order to give a consistent definition of mild solution for equation (1.3) based on an operator theoretical approach, we introduce the following definition (see [20] for a recent discussion about the concept of mild solutions for nonlinear equations and [26] for the approach that we will use in this paper).

Definition 2.1. *Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X. We call A the generator of an (α, β, γ)-regularized family $\{R(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ if the following conditions are satisfied:*
(R1) $R(t)$ is strongly continuous on $\mathbb{R}_+$ and $R(0) = 0$;
(R2) $R(t)D(A) \subset D(A)$ and $AR(t)x = R(t)Ax$ for all $x \in D(A), t \geq 0$;
(R3) The following equation holds:

$$R(t)x = k(t)x + \int_0^t a(t-s)R(s)Axds \tag{2.1}$$

for all $x \in D(A), t \geq 0$. In this case, $R(t)$ is called the (α, β, γ)-regularized family generated by A.

Remark 2.2. It is proved in [23], in the more general context of (a, k)- regularized families, that an operator A is the generator of an (α, β, γ)-regularized family if and only if there exists $\omega \geq 0$ and a strongly continuous function $R : \mathbb{R}_+ \to \mathcal{B}(X)$ such that $\{\frac{\lambda^2+\alpha\lambda^3}{\beta+\gamma\lambda} : Re\lambda > \omega\} \subset \rho(A)$ and

$$H(\lambda)x := \frac{1}{\beta+\gamma\lambda}\left(\frac{\lambda^2+\alpha\lambda^3}{\beta+\gamma\lambda} - A\right)^{-1} x = \int_0^\infty e^{-\lambda t}R(t)xdt, \quad Re\lambda > \omega, \quad x \in X.$$

Because of the uniqueness of the Laplace transform, we note that an (α, β, γ)-regularized family exactly corresponds to an (a, k)-regularized family studied in

[23], with a and k defined at the beginning of this section. In fact, we have

$$\hat{a}(\lambda) = \frac{\beta + \gamma\lambda}{\lambda^2 + \alpha\lambda^3} \quad \text{and} \quad \hat{k}(\lambda) = \frac{1}{\lambda^2 + \alpha\lambda^3}, \quad \text{for all} \quad Re\lambda > \omega.$$

As in the situation of semigroup theory, we have diverse relations of an (α, β, γ)-regularized family and its generator. The following result is a direct consequence of [23, Proposition 3.1 and Lemma 2.2] (see also [11]).

Proposition 2.3. *Let $R(t)$ be an (α, β, γ)-regularized family on X with generator A. Then the following holds:*

(a)For all $x \in D(A)$ we have $R(\cdot)x \in C^2(\mathbb{R}_+; X)$.

(b) Let $x \in X$ and $t \geq 0$. Then $\int_0^t a(t-s)R(s)xds \in D(A)$ and

$$R(t)x = k(t)x + A\int_0^t a(t-s)R(s)xds.$$

Results on perturbation, approximation, asymptotic behavior, representation as well as ergodic type theorems for (α, β, γ)-regularized families can be also deduced from the more general context of $(a, k)-$regularized families (see [22, 23, 24, 25] and [29]).

We recall the following result which provide a wide class of generators of (α, β, γ)-regularized families.

Theorem 2.4 ([14])**.** *Let $-B$ be a positive selfadjoint operator on a Hilbert space H such that*

$$\alpha\beta \leq \gamma\,.$$

Then B is the generator of a bounded (α, β, γ)-regularized family on H.

Let us recall the notion of almost automorphic and asymptotically almost automorphic which shall come into play later on.

Definition 2.5. *A continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that*

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$f(t) = \lim_{n\to\infty} g(t - s_n), \qquad \textit{for each } t \in \mathbb{R}.$$

If the convergence above is uniform in $t \in \mathbb{R}$, then f is almost periodic in the classical Bochner's sense.

Almost automorphic, as a generalization of the classical concept of an almost periodic function, was introduced in the literature by S. Bochner and recently studied by several authors, including [4, 5, 12] among others. A complete description of their properties and further applications to evolution equations can be found in the monographs [18] and [19] by G. M. N'Guérékata.

We remark that the set of all almost automorphic functions, denoted by $AA(X)$, endowed with the sup norm is a Banach space. We define the set $AA(\mathbb{R} \times X; X)$ which consists of all functions $f : \mathbb{R} \times X \to X$ such that $f(\cdot, x) \in AA(X)$ uniformly for each $x \in K$, where K is any bounded subset of X.

Let $C_0(\mathbb{R}_+, X)$ be the subspace of $BC(\mathbb{R}_+, X)$ such that $\lim_{t\to\infty} \|x(t)\| = 0$ and $C_0(\mathbb{R}_+ \times Y, X)$ denotes the space of all continuous functions $h : \mathbb{R}_+ \times Y \to X$ such that $\lim_{t\to\infty} h(t, x) = 0$ uniformly for x in any compact subset of Y.

Definition 2.6. *A continuous function $f : \mathbb{R}_+ \to X$ (resp., $\mathbb{R}_+ \times Y \to X$) is called asymptotically almost automorphic if it admits a decomposition $f = g+\phi$, where $g \in AA(X)$ (resp., $g \in AA(\mathbb{R} \times Y, X)$) and $\phi \in C_0(\mathbb{R}_+, X)$ (resp., $\phi \in C_0(\mathbb{R}_+ \times Y, X)$). Denote by $AAA(X)$ (resp., $AAA(\mathbb{R}_+ \times Y, X)$) the set all such functions.*

We observe that $AAA(X)$ is a Banach space with the sup norm. The next lemma will be very useful for our results.

Lemma 2.7. [27] *Let X and Y be Banach spaces. Suppose that $f \in AAA(\mathbb{R}\times Y; X)$ and g are uniformly continuous on any bounded subset $K \subset Y$, uniformly for $t \geq 0$, where $f = g + h$ with $g \in AA(\mathbb{R} \times Y; X)$ and $h \in C_0(\mathbb{R} \times Y; X)$. If $u \in AAA(Y)$ then $f(\cdot, u(\cdot)) \in AAA(X)$.*

Definition 2.8. *Let f in $AAA(X)$, if $f', f'', \dots, f^{(n)}$ exists and $f', f'', \dots f^{(n)}$ in $AAA(X)$. We say f is n times differentiable asymptotically almost automorphic and we denote $AAA^n(X)$ the set of functions n times differentiable asymptotically almost automorphic.*

The set $AAA^n(X)$ is a Banach space with norm

$$\|f\|_{AAA^n(X)} = \sup_{t\in\mathbb{R}} \sum_{i=0}^{n} \|f^{(i)}(t)\|$$

For more details see [28], pages 1316-1317.

3. Asymptotically almost automorphic solutions

Let $\alpha, \beta, \gamma \in (0, \infty)$. Consider the linear equation

$$u''(t) + \alpha u'''(t) = \beta Au(t) + \gamma Au'(t) + f(t), \tag{3.1}$$

with initial conditions $u(0) = x, u'(0) = y, u''(0) = z$, where A is the generator of a (α, β, γ)-regularized family $R(t)$. By a strong solution of (3.1) we understand a function $u \in C(\mathbb{R}_+; D(A)) \cap C^3(\mathbb{R}_+; X)$ such that $u' \in C(\mathbb{R}_+; D(A))$ and verify (3.1).

The following result gives a complete description of the solutions for equation (3.1) in terms of (α, β, γ)-regularized families. It corresponds to an extension of the standard variation of parameters formula for the second order Cauchy problem.

Proposition 3.1. [11, Proposition 3.1] *Let $R(t)$ be an $(\alpha, \beta, \gamma)-$regularized family on X with generator A. If $f \in L^1_{loc}(\mathbb{R}_+, D(A^2))$, $x \in D(A^3)$, $y \in D(A^2)$ and $z \in D(A^2)$ then $u(t)$ given by*

$$u(t) = \alpha R''(t)x + R'(t)x - \gamma AR(t)x + \alpha R'(t)y + R(t)y + \alpha R(t)z + \int_0^t R(t-s)f(s)ds, \quad t \geq 0, \tag{3.2}$$

is a solution of (3.1).

The following assumption was introduced in [11]:

(ED) There are constants $M > 0$ and $\omega > 0$ such that

$$||R'(t)|| + ||R(t)|| \leq Me^{-\omega t}, \quad t \geq 0.$$

We say in short that $R(t)$ and $R'(t)$ are exponentially stable. We introduce the following condition:

$(\mathbf{ED})^*$ There are constants $M > 0$ and $\omega > 0$ such that

$$||R''(t)|| + ||R'(t)|| + ||R(t)|| \leq Me^{-\omega t}, \quad t \geq 0.$$

The following result on regularity of the convolution under asymptotically almost automorphic functions is one of the keys to obtain our results.

Lemma 3.2. *Let $R(t)$ be an exponentially stable (α, β, γ)-regularized family on X with generator A. If $f \in AAA(X)$ then the function*

$$F(t) = \int_0^t R(t-s)f(s)ds$$

belongs to $AAA(X)$.

Proof. If $f = g + h$ with $g \in AA(X)$ and $h \in C_0(\mathbb{R}_+, X)$ then we have that $F(t) = G(t) + H(t)$, where

$$G(t) := \int_{-\infty}^t R(t-s)g(s)ds$$

and

$$H(t) := \int_0^t R(t-s)h(s)ds - \int_{-\infty}^0 R(t-s)g(s)ds.$$

We note that $G(t) \in AA(X)$ by [2, Lemma 3.1], now we claim that $\|H(t)\| \to 0$ as $t \to \infty$. In fact, for each $\epsilon > 0$ there exists a $T > 0$ such that $\|h(s)\| \leq \epsilon$ for all $s > T$. Then for all $t > 2T$ we deduce

$$\begin{aligned} \|H(t)\| &\leq \int_0^{t/2} Me^{-\omega(t-s)}\|h(s)\|ds + \int_{t/2}^t Me^{-\omega(t-s)}\|h(s)\|ds + \int_t^\infty Me^{-\omega s}\|g(t-s)\|ds \\ &\leq M(\|h\|_\infty + \|g\|_\infty)\int_t^\infty e^{-\omega s}ds + \epsilon M\int_0^\infty e^{-\omega s}ds. \end{aligned}$$

Therefore, $\lim_{t\to\infty} H(t) = 0$, that is, $H \in C_0(\mathbb{R}_+, X)$. This completes the proof. □

We begin our results on existence of asymptotically almost automorphic functions for the linear equation, with the following theorem.

Theorem 3.3. *Let $R(t)$ be an (α, β, γ)-regularized family on X with generator A that satisfies assumption* **(ED)**. *If $f \in AAA(X)$ is such that $f(t) \in D(A^2)$ for all $t \geq 0$, then Eq. (3.1) with initial conditions $u(0) = 0$, $u'(0) = y \in D(A^2)$ and $u''(0) = z \in D(A^2)$ has a unique strong solution $u \in AAA(X)$.*

Proof. Let $f \in AAA(X)$ such that $f(t) \in D(A^2)$ and $y, z \in D(A^2)$. From Proposition 3.1 we have that the solution for Eq. (3.1) is given by

$$u(t) = \alpha R'(t)y + R(t)y + \alpha R(t)z + \int_0^t R(t-s)f(s)ds.$$

From Lemma 3.2 we have that $g(t) = \int_0^t R(t-s)f(s)ds$ belongs to $AA(X)$. On the other hand, if $t \to \infty$ we have that $\|\alpha R'(t)y\| \leq \alpha\|y\|Me^{-\omega t} \to 0$, $\|R(t)y\| \leq \|y\|Me^{-\omega t} \to 0$ and

$$\|\alpha R(t)z\| \leq \alpha\|z\|Me^{-\omega t} \to 0.$$

Therefore, $u \in AAA(X)$. □

From now we study the semilinear version of Eq. (3.1). We consider first the initial value problem

$$\text{(3.3)} \qquad \begin{cases} u''(t) + \alpha u'''(t) = \beta Au(t) + \gamma Au'(t) + f(t, u(t)), \quad t \geq 0; \\ \\ u(0) = 0, u'(0) = y, u''(0) = z, \end{cases}$$

where $\alpha, \beta, \gamma \in (0, \infty)$, A is the generator of a (α, β, γ)-regularized family $R(t)$ and $f : \mathbb{R}_+ \times X \to X$ is a suitable function.

Definition 3.4. [11] *Let $R(t)$ be an (α, β, γ)-generalized family on X with generator A. A continuous function $u : \mathbb{R}_+ \to X$ satisfying the integral equation*

$$u(t) \;=\; \alpha R'(t)y + R(t)y + \alpha R(t)z + \int_0^t R(t-s)f(s, u(s))ds, \;\forall\; t \geq 0,$$

where $y, z \in X$ is called a mild solution to the equation (3.3).

We study conditions to existence and uniqueness of a mild solution for equation (3.3) when the function f is Lipschitz continuous.

Theorem 3.5. *Let $R(t)$ be an (α,β,γ)-regularized family on X with generator A that satisfies assumption* **(ED)**. *Let $f \in AAA(\mathbb{R}_+ \times X, X)$ and suppose that there exists an integrable bounded function $L : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|f(t,x) - f(t,y)\| \leq L(t)\|x-y\|, \ \forall x,y \in X, \ t \geq 0.$$

Then equation (3.3) has a unique asymptotically almost automorphic mild solution.

Proof. We define the operator Λ on the space $AAA(X)$ by

$$\Lambda u(t) = \alpha R'(t)y + R(t)y + \alpha R(t)z + \int_0^t R(t-s)f(s,u(s))ds.$$

We show that $\Lambda u \in AAA(X)$. Initially we observe that since $R(t)y \to 0$, $\alpha R(t)z \to 0$ and $\alpha R'(t)y \to 0$ as $t \to \infty$, then $R(\cdot)y\,, \alpha R(\cdot)z$ and $\alpha R'(\cdot)y \in C_0(X)$. It follows from Lemma 2.7 that the function $s \to f(s,u(s))$ is asymptotically almost automorphic; then by Lemma 3.2

$$\int_0^t R(t-s)f(s,u(s))ds \in AAA(X).$$

Furthermore, for $u_1, u_2 \in AAA(X)$ we have that

$$\begin{aligned} \|\Lambda u_1(t) - \Lambda u_2(t)\| &\leq M \int_0^t e^{-\omega(t-s)} L(s)ds\|u_1-u_2\|_\infty \\ &\leq M \int_0^t L(s)ds\|u_1-u_2\|_\infty \\ &\leq M\|L\|_1\|u_1-u_2\|_\infty, \end{aligned}$$

hence,

$$\begin{aligned} \|(\Lambda^2 u_1)(t) - (\Lambda^2 u_2)(t)\| &\leq M^2 \left(\int_0^t L(s)\left(\int_0^s L(\tau)d\tau\right)ds\right)\|u_1-u_2\|_\infty \\ &\leq \frac{M^2}{2}\left(\int_0^t L(\tau)d\tau\right)^2\|u_1-u_2\|_\infty \\ &\leq \frac{(M\|L\|_1)^2}{2}\|u_1-u_2\|_\infty. \end{aligned}$$

In general, we get the following estimate

$$\|(\Lambda^n u_1)(t) - (\Lambda^n u_2)(t)\| \leq \frac{(C\|L\|_1)^n}{n!}\|u_1-u_2\|_\infty.$$

Since $\frac{(M\|L\|_1)^n}{n!} < 1$ for n sufficiently large, by the fixed point iteration method Λ has a unique fixed point $u \in AAA(X)$. This completes the proof. $\square$

In what follows we study the semilinear equation

$$(3.4)\qquad \begin{cases} u''(t)+\alpha u'''(t)=\beta Au(t)+\gamma Au'(t)+f(t,u(t),u'(t)), & t\geq 0;\\ \\ u(0)=0, u'(0)=0, u''(0)=z, & . \end{cases}$$

where $\alpha,\beta,\gamma\in(0,\infty)$, A is the generator of a (α,β,γ)-regularized family $R(t)$ and $f:\mathbb{R}_+\times X\times X\to X$ is a suitable function.

The appropriate concept of mild solution reads now as follows.

Definition 3.6. *Let $R(t)$ be an (α,β,γ)-generalized family on X with generator A. A continuous function $u:\mathbb{R}_+\to X$ satisfying the integral equation*

$$u(t)\ =\ \alpha R(t)z+\int_0^t R(t-s)f(s,u(s),u'(s))ds,\ \forall\ t\geq 0,$$

where $z\in D(A)$ is called a mild solution to the equation (3.4).

To study existence of almost automorphic mild solutions, we will need the following result.

Lemma 3.7. *Let X and Y be Banach spaces. Suppose that $f\in AAA(\mathbb{R}\times X\times X;X)$ and g is uniformly continuous on any bounded subset $K\subset X$, uniformly for $t\geq 0$, where $f=g+h$ with $g\in AA(\mathbb{R}\times X\times X;X)$ and $h\in C_0(\mathbb{R}\times X\times X;X))$. If $x(\cdot),y(\cdot)\in AAA(X)$ then $f(\cdot,x(\cdot),y(\cdot))\in AAA(X)$.*

Proof. Let $x(\cdot),y(\cdot)\in AAA(X)$, then $x(t)=\alpha_1(t)+\beta_1(t)$, $y(t)=\alpha_2(t)+\beta_2(t)$, where $\alpha_1(\cdot),\alpha_2(\cdot)\in AA(\mathbb{R},X)$ and $\beta_1(\cdot),\beta_2(\cdot)\in C_0(\mathbb{R},X)$. Then

$$\begin{aligned} f(t,x(t),y(t)) &= g(t,\alpha_1(t),\alpha_2(t))+f(t,x(t),y(t))-g(t,\alpha_1(t),\alpha_2(t))\\ &= g(t,\alpha_1(t),\alpha_2(t))+g(t,x(t),y(t))-g(t,\alpha_1(t),\alpha_2(t))+h(t,x(t),y(t)). \end{aligned}$$

We will prove $g(\cdot,\alpha_1(\cdot),\alpha_2(\cdot))\in AA(\mathbb{R}\times X\times X;X)$ and $f(\cdot,x(\cdot),y(\cdot))-g(\cdot,\alpha_1(\cdot),\alpha_2(\cdot))\in C_0(\mathbb{R}\times X\times X,X)$.

First, we will prove $g(\cdot,\alpha_1(\cdot),\alpha_2(\cdot))\in AA(\mathbb{R}\times X\times X;X)$. Since $g(\cdot,x,y),\alpha_1(\cdot),\alpha_2(\cdot)$ are almost authomorphic in t, then for any sequence $(t_n')\subset\mathbb{R}$, there exist a subsequence $(t_n)\subset(t_n)$ such that

$$\begin{aligned} \lim_{n\to\infty} g(t+t_n,x,y) &= \overline{g}(t,x,y)\quad\text{and}\lim_{n\to\infty}\overline{g}(t-t_n,x,y)=g(t,x,y),\ \forall\,x,y\in X.\\ \lim_{n\to\infty}\alpha_1(t+t_n) &= \overline{\alpha}_1(t)\quad\text{and}\lim_{n\to\infty}\overline{\alpha}_1(t-t_n)=\alpha_1(t)\\ \lim_{n\to\infty}\alpha_2(t+t_n) &= \overline{\alpha}_2(t)\quad\text{and}\lim_{n\to\infty}\overline{\alpha}_2(t-t_n)=\alpha_2(t) \end{aligned}$$

for all $t\in\mathbb{R}$. By the other hand, we have

$$\|g(t+t_n,\alpha_1(t+t_n),\alpha_2(t+t_n))-\overline{g}(t,\overline{\alpha}_1(t),\overline{\alpha}_2(t))\|$$

$$\leq \|g(t+t_n,\alpha_1(t+t_n),\alpha_2(t+t_n))-g(t+t_n,\overline{\alpha}_1(t),\overline{\alpha}_2(t))\|+\|g(t+t_n,\overline{\alpha}_1(t),\overline{\alpha}_2(t))-\overline{g}(t,\overline{\alpha}_1(t),\overline{\alpha}_2(t))\|$$

Since $\alpha_1, \alpha_2, \overline{\alpha}_1, \overline{\alpha}_2$ are bounded functions, there exist a bounded set $K \subset X$, such that

$$\alpha_1(t), \alpha_2(t), \overline{\alpha}_1(t), \overline{\alpha}_2(t) \in K \,\forall\, t \in \mathbb{R}.$$

Then

$$\lim_{n\to\infty} \|g(t+t_n,\alpha_1(t+t_n),\alpha_2(t+t_n)) - g(t+t_n,\overline{\alpha}_1(t),\overline{\alpha}_2(t))\| = 0.$$

Therefore

$$\lim_{n\to\infty} \|g(t+t_n,\alpha_1(t+t_n),\alpha_2(t+t_n)) - \overline{g}(t,\overline{\alpha}_1(t),\overline{\alpha}_2(t))\| = 0.$$

In similar way, we prove

$$\lim_{n\to\infty} \|\overline{g}(t-t_n,\overline{\alpha}_1(t-t_n),\overline{\alpha}_2(t-t_n)) - g(t,\alpha_1(t),\alpha_2(t))\| = 0,$$

using the inequality

$$\|\overline{g}(t-t_n,\overline{\alpha}_1(t-t_n),\overline{\alpha}_2(t-t_n)) - g(t,\alpha_1(t),\alpha_2(t))\|$$

$$\leq \|\overline{g}(t-t_n,\overline{\alpha}_1(t-t_n),\overline{\alpha}_2(t-t_n))-g(t-t_n,\alpha_1(t),\alpha_2(t))\|+\|g(t-t_n,\alpha_1(t),\alpha_2(t))-g(t,\alpha_1(t),\alpha_2(t))\|.$$

Now we will prove the second part. Note that

$$\begin{aligned} \|x(t)-\alpha_1(t)\| &= \|\beta_1(t)\| \\ \|y(t)-\alpha_2(t)\| &= \|\beta_2(t)\|. \end{aligned}$$

Then for all $\varepsilon > 0$, there exists $T > 0$, such that

$$\begin{aligned} \|x(t)-\alpha_1(t)\| &< \varepsilon \quad \text{and} \\ \|y(t)-\alpha_2(t)\| &< \varepsilon, \quad \forall t \geq T, \end{aligned}$$

Since g is uniformly continuous on any bounded subset $K \subset X$, uniformly for $t \geq 0$, we have for all $\varepsilon' > 0$, there exists $\delta > 0$, such that $\|x - x'\| < \delta$, $\|y - y'\| < \delta$, and $x, x', y, y' \in K$, where K is any bounded subset of X then

$$\|g(t,x,y) - g(t,x',y')\| < \varepsilon'.$$

In particular, we take $\varepsilon = \delta$, then

$$\begin{aligned} \|x(t)-\alpha_1(t)\| &< \delta \quad \text{and} \\ \|y(t)-\alpha_2(t)\| &< \delta, \quad \forall t \geq T. \end{aligned}$$

Furthemore $x(\cdot), \alpha_1(\cdot), y(\cdot), \alpha_2(\cdot)$ are bounded functions, then exists a bounded set $K \subset X$, such that $x(t), \alpha_1(t), y(t), \alpha_2(t) \in K,$ for all $t \in \mathbb{R}$. Then, we have

$$\|g(t,x(t),y(t)) - g(t,\alpha_1(t),\alpha_2(t))\| < \varepsilon', \quad \forall t \geq T.$$

Then

$$\lim_{t\to\infty} \|g(t,x(t),y(t)) - g(t,\alpha_1(t),\alpha_2(t))\| = 0,$$

furthermore

$$\lim_{t\to\infty} \|h(t,x(t),y(t))\| = 0.$$

Therefore $g(\cdot,x(\cdot),y(\cdot)) - g(\cdot,\alpha_1(t\cdot),\alpha_2(\cdot)) + h(\cdot,x(\cdot),y(\cdot)) \in C_0(\mathbb{R}\times X\times X,X)$. $\square$

Following the same lines of the proof of Lemma 3.7, we get the following result for the space $AAA^2(\mathbb{R}\times X\times X\times X,X)$.

Lemma 3.8. *Let X and Y be Banach spaces. Suppose that $f \in AAA(\mathbb{R}\times X\times X\times X;X)$ and g is uniformly continuous on any bounded subset $K \subset X$, uniformly for $t \geq 0$, where $f = g + h$ with $g \in AA(\mathbb{R}\times X\times X\times X;X)$ and $h \in C_0(\mathbb{R}\times X\times X\times X;X))$. If $x(\cdot),y(\cdot),z(\cdot) \in AAA(X)$ then $f(\cdot,x(\cdot),y(\cdot),z(\cdot)) \in AAA(X)$.*

The following is our second main result in this paper.

Theorem 3.9. *Let $R(t)$ be an (α,β,γ)-regularized family on X with generator A that satisfies assumption* **(ED)**. *Let $f \in AAA(\mathbb{R}_+\times X\times X,X)$ and suppose there exists constants L_1,L_2 such that $\max\{L_1,L_2\} < \frac{2w}{M}$ and*

$$\|f(t,x,x') - f(t,y,y')\| \leq L_1\|x-y\| + L_2\|x'-y'\|,\ \forall x,y,x',y' \in X,\ t\geq 0.$$

Then equation (3.4) has a unique differentiable asymptotically almost automorphic mild solution.

Proof. We define the operator Λ on the space $AAA^1(X)$ by

$$\Lambda u(t) = \alpha R(t)z + \int_0^t R(t-s)f(s,u(s),u'(s))ds.$$

By Lemma 3.7 $\int_0^t R(t-s)f(s,u(s),u'(s))ds \in AA(X)$ and $R(\cdot) \in C_0(\mathbb{R}_+,X)$, then $\Lambda u(t) \in AAA(X)$. Furthermore, using $R(0) = 0$ and $z \in D(A)$, we have

$$(\Lambda u)'(t) = \alpha R'(t)z + \int_0^t R'(t-s)f(s,u(s),u'(s))ds. \tag{3.5}$$

By Lemma 3.7 $\int_0^t R'(t-s)f(s,u(s),u'(s))ds \in AA(X)$ and $R'(\cdot) \in C_0(\mathbb{R}_+,X)$, then $(\Lambda u)'(t) \in AAA(X)$. Therefore $\Lambda : AAA^1(X) \to AAA^1(X)$ is well defined and, for

$u_1, u_2 \in AAA^1(X)$ we have that

$$\begin{aligned}
\|\Lambda u_1 - \Lambda u_2\|_\infty &\leq \sup_{t\in\mathbb{R}_+} \int_0^t \|R(t-s)\| \|f(s, u_1(s), u_1'(s)) - f(s, u_2(s), u_2'(s))\| \, ds \\
&\leq \sup_{t\in\mathbb{R}_+} \int_0^t \|R(t-s)\| [L_1 \|u_1(s) - u_2(s)\| + L_2 \|u_1'(s) - u_2'(s)\|] \, ds \\
&\leq M[L_1 \|u_1 - u_2\|_\infty + L_2 \|u_1' - u_2'\|_\infty] \sup_{t\in\mathbb{R}_+} \int_0^t e^{-w(t-s)} \, ds. \\
&= ML[\|u_1 - u_2\|_\infty + \|u_1' - u_2'\|_\infty] \frac{1}{w} \\
&= ML \frac{1}{w} \|u_1 - u_2\|_{AAA^1(X)}.
\end{aligned}$$

where $L = \max\{L_1, L_2\}$. In similar way, we have

$$\begin{aligned}
\|(\Lambda u_1)' - (\Lambda u_2)'\|_\infty &\leq \sup_{t\in\mathbb{R}_+} \int_0^t \|R'(t-s)\| \|f(s, u_1(s), u_1'(s)) - f(s, u_2(s), u_2'(s))\| \, ds \\
&\leq \sup_{t\in\mathbb{R}_+} \int_0^t \|R'(t-s)\| [L_1 \|u_1(s) - u_2(s)\| + L_2 \|u_1'(s) - u_2'(s)\|] \, ds \\
&\leq M[L_1 \|u_1 - u_2\|_\infty + L_2 \|u_1' - u_2'\|_\infty] \sup_{t\in\mathbb{R}_+} \int_0^t e^{-w(t-s)} \, ds. \\
&= ML[\|u_1 - u_2\|_\infty + \|u_1' - u_2'\|_\infty] \frac{1}{w} \\
&= ML \frac{1}{w} \|u_1 - u_2\|_{AAA^1(X)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|\Lambda u_1 - \Lambda u_2\|_{AAA^1(X)} &= \|\Lambda u_1 - \Lambda u_2\|_\infty + \|(\Lambda u_1)' - (\Lambda u_2)'\|_\infty \\
&\leq ML \frac{2}{w} \|u_1 - u_2\|_{AAA^1(X)}.
\end{aligned}$$

This proves that Λ is a contraction, so by the Banach fixed point theorem there exists a unique $u \in AAA^1(X)$ such that $\Lambda u = u$, proving the theorem. $\square$

We have a similar result using the condition $(\mathbf{ED}^*)$.

Theorem 3.10. *Let $R(t)$ be an (α, β, γ)-regularized family on X with generator A that satisfies assumption $(\mathbf{ED})^*$. Let $f \in AAA(\mathbb{R}_+ \times X \times X, X)$ and suppose there exists constants L_1, L_2 such that $\max\{L_1, L_2\} < \frac{2w}{M}$ and*

$$\|f(t, x, x') - f(t, y, y')\| \leq L_1 \|x - y\| + L_2 \|x' - y'\|, \ \forall x, y, x', y' \in X, \ t \geq 0.$$

Then equation (3.4) has a unique differentiable asymptotically almost automorphic mild solution.

Similarly we have the following result for the space $AAA^2(\mathbb{R}_+ \times X \times X \times X, X)$.

Theorem 3.11. *Let $R(t)$ be an (α, β, γ)-regularized family on X with generator A that satisfies assumption $(\boldsymbol{ED}^*)$. Let $f \in AAA(\mathbb{R}_+ \times X \times X \times X, X)$ and suppose there exists constants L_1, L_2, L_3 such that* $\max\{L_1, L_2, L_3\} < \dfrac{3w}{M}$ *and*

(3.6)
$$\|f(t,x,x',x'')-f(t,y,y,y'')\| \leq L_1\|x-y\|+L_2\|x'-y'\|+L_3\|x''-y''\|,\ \forall\, x,y,x',y',x'',y'' \in X, t \geq 0$$

Then equation (3.4) has a unique twice differentiable asymptotically almost automorphic mild solution.

Proof. The proof is similar to the proof of Theorem 3.9, using the inequality

$$\begin{aligned} ||\Lambda u_1 - \Lambda u_2||_{AA^2(X)} &= ||\Lambda u_1 - \Lambda u_2||_\infty + ||(\Lambda u_1)' - (\Lambda u_2)'||_\infty + ||(\Lambda u_1)'' - (\Lambda u_2)''||_\infty \\ &\leq ML\frac{3}{\omega}||u_1 - u_2||_{AA^2(X)}. \end{aligned}$$

□

4. Existence of mild solutions with nonlocal conditions

In this section, we use the Hausdorff measure of noncompactness and a fixed point argument to prove the existence of a mild solution for an special case of equation (3.4) with a nonlocal initial condition. More precisely, we consider

$$(4.7)\qquad \begin{cases} u''(t) + \alpha u'''(t) = \beta Au(t) + \gamma Au'(t) + f(t,u(t)), & t \in I := [0,1]; \\ \\ u(0) = 0, u'(0) = 0, u''(0) = g(u), & . \end{cases}$$

where A is the generator of a (α, β, γ)-regularized family $R(t)$ and $f : I \times X \to X$, $g : C([0,1]; X) \to X$ are suitable functions.

In order to give our main result, we consider the following assertions

(H1) A generates a norm continuous (for $t > 0$) (α, β, γ)-regularized family $R(t)$. We denote $M = \sup\{||R(t)|| : t \in [0,1]\}$.

(H2) $g : C([0,1]; X) \to X$ is continuous and compact, there exists positive constants c and d such that $\|g(u)\| \leqslant c\|u\| + d$, $\forall u \in C([0,1]; X)$.

(H3) $f : [0,1] \times X \to X$ satisfies the Carathéodory type conditions, that is, $f(\cdot, x)$ is measurable for all $x \in X$ and $f(t, \cdot)$ is continuous for almost all $t \in [0,1]$.

(H4) There exists a function $m \in L^1(0,1;\mathbb{R}^+)$ and a nondecreasing continuous function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that
$$\|f(t,x)\| \leqslant m(t)\Phi(\|x\|)$$
for all $x \in X$ and almost all $t \in [0,1]$.

(H5) There exists a function $H \in L^1(0,1;\mathbb{R}^+)$ such that for any bounded $B \subseteq X$

$$\gamma(f(t,B)) \leqslant H(t)\gamma(B)$$

for almost all $t \in [0,1]$.

In $(H5)$ γ denote the Hausdorff measure of noncompactness which is defined by

$$\gamma(B) = \inf\{\epsilon > 0 : B \text{ has a finite cover by balls of radius } \epsilon\}.$$

We note that this measure of noncompactness satisfies interesting regularity properties. For more information, we refer to [3]. We are now in position to establish the following result.

Theorem 4.12. *If the hypothesis (H1)-(H5) are satisfied and there exists a constant $R > 0$ such that*

$$M(cR+d) + M\Phi(R)\int_0^1 m(s)ds \leqslant R$$

then the problem (4.7) *has at least one mild solution.*

Proof. Define $F : C([0,1];X) \to C([0,1];X)$ by

$$(Fx)(t) = R(t)g(x) + \int_0^t R(t-s)f(s,x(s))ds, \quad t \in [0,1]$$

for all $x \in C([0,1];X)$. First we show that F is a continuous map.
Let $\{x_n\}_{n\in\mathbb{N}} \subseteq C([0,1];X)$ such $x_n \to x$ (in the norm of $C([0,1];X)$) Note that

$$\|F(x_n) - F(x)\| \leqslant M\|g(x_n) - g(x)\| + M\int_0^1 \|f(s,x_n(s)) - f(s,x(s))\|ds, \tag{4.8}$$

by (H2) and (H3) and dominated convergence theorem we conclude that $\|F(x_n) - F(x)\| \to 0$ when $n \to \infty$.
Now denote by $B_R := \{x \in C([0,1];X) : \|x(t)\| \leqslant R \text{ for all } t \in [0,1]\}$ and note that for any $x \in B_R$ we have

$$\begin{aligned}\|(Fx)(t)\| &\leqslant \|R(t)g(x)\| + \left\|\int_0^t R(t-s)f(s,x(s))ds\right\| \\ &\leqslant M(cR+d) + M\Phi(R)\int_0^1 m(s)ds \leqslant R\end{aligned}$$

Therefore $F : B_R \to B_R$ and $F(B_R)$ is a bounded set. Moreover, by continuity of function $t \to R(t)$ we have that $F(B_R)$ is an equicontinuous set of functions. Define $B := \overline{co}(F(B_R))$. Then B is an equicontinuous set and $F : B \to B$ is a bounded

continuous operator.
Let $\varepsilon > 0$. By [31, Lemma 2.4] there exists $\{y_n\}_{n\in\mathbb{N}} \subset F(B)$ such that

$$\begin{array}{rcl} \gamma(FB(t)) & \leqslant & 2\gamma(\{y_n(t)\}_{n\in\mathbb{N}}) + \varepsilon \\ & \leqslant & 2\gamma\left(\int_0^t R(t-s)f(s,\{x_n(s)\}_{n\in\mathbb{N}})ds\right) + \varepsilon \\ & \leqslant & 4M\int_0^t \gamma(f(s,\{x_n(s))\}_{n\in\mathbb{N}})ds + \varepsilon \\ & \leqslant & 4M\int_0^t H(s)\gamma(\{x_n(s)\}_{n\in\mathbb{N}})ds + \varepsilon \\ & \leqslant & 4M\gamma(\{x_n\})\int_0^t H(s)ds + \varepsilon \\ & \leqslant & 4M\gamma(B)\int_0^t H(s)ds + \varepsilon \end{array} \tag{4.9}$$

Since $H \in L^1(0,1;X)$ there exists $\varphi \in C([0,1];\mathbb{R}_+)$ such that $\displaystyle\int_0^1 |H(s)-\varphi(s)|ds <$ $\alpha \;\; \left(\alpha < \dfrac{1}{4M}\right)$. Therefore

$$\begin{aligned} \gamma(FB(t)) &\leqslant 4M\gamma(B)\left[\int_0^t |H(s)-\varphi(s)|ds + \int_0^t \varphi(s)ds\right] + \varepsilon \\ &\leqslant 4M\gamma(B)\left[\gamma + Nt\right] + \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary we obtain that

$$\gamma(FB(t)) \leqslant (a+bt)\gamma(B) \tag{4.10}$$

where $a = 4\gamma M$ and $b = 4MN$.

Let $\varepsilon > 0$, by [31, Lemma 2.4] there exists $\{y_n\}_{n\in\mathbb{N}} \subseteq \overline{co}(F(B))$ such that

$$
\begin{aligned}
\gamma(F^2(B(t))) &\leqslant 2\gamma\left(\int_0^t R(t-s)f(s,\{y_n(s)\}_{n\in\mathbb{N}})ds\right)+\varepsilon\\
&\leqslant 4M\int_0^t \gamma(f(s,\{y_n(s)\}_{n\in\mathbb{N}}))ds+\varepsilon\\
&\leqslant 4M\int_0^t H(s)\gamma(\overline{co}F^1B(s))+\varepsilon\\
&\leqslant 4M\int_0^t H(s)\gamma(F^1B(s))+\varepsilon\\
&\leqslant 4M\int_0^t \big[|H(s)-\varphi(s)|+|\varphi(s)](a+bs)\gamma(B)ds+\varepsilon\\
&\leqslant 4M(a+bt)\int_0^t |H(s)-\varphi(s)|ds+4MN(at+\frac{bt^2}{2})+\varepsilon\\
&\leqslant a(a+bt)+b(at+\frac{bt^2}{2})+\varepsilon
\end{aligned}
$$

Since $\varepsilon>0$ is arbitrary then

$$\gamma(F^2(B(t)))\leqslant\left(a^2+2bt+\frac{(bt)^2}{2}\right)\gamma(B) \tag{4.11}$$

By an iterative process we obtain

$$\gamma(F^n(B(t)))\leqslant\left(a^n+C_n^1a^{n-1}bt+C_n^2a^{n-2}\frac{(bt)^2}{2!}+\cdots+\frac{(bt)^n}{n!}\right)\gamma(B) \tag{4.12}$$

By [31, Lemma 2.1] we obtain that

$$\gamma(F^n(B))\leqslant\left(a^n+C_n^1a^{n-1}b+C_n^2a^{n-2}\frac{b^2}{2!}+\cdots+\frac{b^n}{n!}\right)\gamma(B) \tag{4.13}$$

From [31, Lemma 2.5] we know that there exists $n_0\in\mathbb{N}$ such that

$$\left(a^{n_0}+C_{n_0}^1a^{n_0-1}b+C_{n_0}^2a^{n_0-2}\frac{b^2}{2!}+\cdots+\frac{b^{n_0}}{n_0!}\right)=r<1 \tag{4.14}$$

With this we conclude that

$$\gamma(F^{n_0}B)\leqslant r\gamma(B) \tag{4.15}$$

By [31, Lemma 2.6] , F has a fixed point in B, and this fixed point is a mild solution of equation (4.7). □

We finish this paper with the following application.

Example 4.13. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and $0 < \lambda < \mu$. Consider the equation* [6]*:*

$$u''(t) + \lambda u'''(t) = c^2\Delta u(t) + c^2\mu\Delta u'(t) + f(t, u(t)). \tag{4.16}$$

Define $\alpha = \lambda, \beta = c^2$ and $\gamma = c^2\mu$. Then $\alpha\beta < \gamma$. It is well known that the Dirichlet-Laplacian operator Δ with domain $H^2(\Omega) \cap H_0^1(\Omega)$ is a selfadjoint operator on $L^2(\Omega)$ and $\sigma(\Delta) \subset (-\infty, 0)$. It follows from Theorem 2.4 that $B = \Delta$ is the generator of an (α, β, γ)-regularized family $R(t)$ on $X = L^2(\Omega)$. By Proposition 3.1 it follows that $u(t) = \alpha R(t)w$ is the unique solution of (4.16) with initial conditions $u(0) = u'(0) = 0$ and $u''(0) = w \in D(\Delta^2)$.

Define $v = u + \lambda u'$. Then the equation (4.16) becomes the system

$$v''(t) = c^2\Delta v(t) + c^2(\mu - \lambda)\Delta u'(t) \tag{4.17}$$

with initial condition $v(0) = 0$. The energy functional of this system is given by

$$E(t) = \frac{1}{2}\int_\Omega v'^2 + c^2|\nabla v|^2 + c^2\lambda(\nu - \lambda)|\nabla u'|^2 dx. \tag{4.18}$$

It was proved by Bose and Gorain [6] *that the energy of the system tends to zero exponentially as $t \to \infty$, that is, there exists constants $M > 0$ and $\nu > 0$ such that*

$$E(t) \leq Me^{-\nu t}, \quad t \geq 0. \tag{4.19}$$

In particular, from the definition of $E(t)$ follows that $||\nabla v(t)||_{L^2} \leq Me^{-\nu t}$ and $||\nabla u'(t)||_{L^2} \leq Me^{-\nu t}$. Hence, Poincare's inequality and the definition of v implies that there exists a constant $C_w > 0$ such that

$$||\alpha R(t)w|| = ||u(t)|| \leq ||v(t)|| + \lambda||u'(t)|| \leq C_w e^{-\nu t}$$

In particular, $||R'(t)w|| = ||u'(t)|| \leq C_w e^{-\nu t}$. On the other hand, note that from (3.14) and (3.15) we have

$$||v'(t)||_{L^2} \leq C_w e^{-\nu t}, \quad t \geq 0.$$

Hence,

$$\begin{aligned} ||\lambda R''(t)w||_{L^2} &= ||\lambda u''(t)||_{L^2} \\ &= ||v'(t) - \lambda u'(t)||_{L^2} \\ &\leq ||v'(t)||_{L^2} + ||u'(t)||_{L^2} \\ &\leq M_w e^{-\nu t}. \end{aligned}$$

Finally, the uniform boundedness principle implies that there exists $C > 0$ such that

$$||R''(t)|| + ||R'(t)|| + ||R(t)|| \leq Ce^{-\nu t}, \quad t \geq 0. \tag{4.20}$$

Now, from Theorem 3.3 we conclude that for each $f \in AAA(L^2(\Omega))$ such that $f(t) \in D(\Delta^2)$ for all $t \geq 0$, the linear equation

$$u''(t) + \lambda u'''(t) = c^2\Delta u(t) + c^2\mu\Delta u'(t) + f(t)$$

with initial conditions $u(0) = 0$, $u'(0) = y \in D(\Delta^2)$ *and* $u''(0) = z \in D(\Delta^2)$ *has a unique solution* $u \in AAA(L^2(\Omega))$. *On the other hand, from Theorem 3.5, given* $f \in AAA(\mathbb{R}_+ \times X, X)$ *and assuming that there exists an integrable bounded function* $L : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\|f(t,x) - f(t,y)\| \le L(t)\|x - y\|, \ \forall x, y \in X, \ t \ge 0, \tag{4.21}$$

we conclude that the nonlinear equation (4.16) with initial conditions $u(0) = 0$ *and* $u'(0) = y \in L^2(\Omega)$ *and* $u''(0) = z \in L^2(\Omega)$ *has a unique asymptotically almost automorphic mild solution.*

Now consider the equation

$$u''(t) + \lambda u'''(t) = c^2\Delta u(t) + c^2\mu\Delta u'(t) + f(t, u(t), u'(t)), \tag{4.22}$$

with initial conditions $u(0) = 0$ *and* $u'(0) = y \in L^2(\Omega)$ *and* $u''(0) = z \in L^2(\Omega)$. *From Theorem 3.10, given* f *in* $AAA(\mathbb{R}_+ \times X \times X, X)$ *and assuming there exists constants* L_1, L_2 *such that* $\max\{L_1, L_2\} < \dfrac{2w}{M}$ *and*

$$\|f(t,x,x') - f(t,y,y')\| \le L_1\|x - y\| + L_2\|x' - y'\|, \ \forall x, y, x', y' \in X, \ t \ge 0,$$

we obtain that the equation (4.22) has a unique differentiable asymptotically almost automorphic mild solution. Finally for the equation

$$u''(t) + \lambda u'''(t) = c^2\Delta u(t) + c^2\mu\Delta u'(t) + f(t, u(t), u'(t), u''(t)), \tag{4.23}$$

with initial conditions $u(0) = u'(0) = 0$ *and* $u''(0) = z \in L^2(\Omega)$. *From Theorem 3.11, given* $f \in AAA(\mathbb{R}_+ \times X \times X \times X, X)$ *and assuming there exists constants* L_1, L_2, L_3 *such that* $\max\{L_1, L_2, L_3\} < \dfrac{3w}{M}$ *and*

$$\|f(t,x,x',x'') - f(t,y,y,y'')\| \le L_1\|x-y\| + L_2\|x'-y'\| + L_3\|x''-y''\|, \ \forall\, x, y, x', y', x'', y'' \in X, t \ge 0, \tag{4.24}$$

we obtain that the equation (4.23) has a unique twice differentiable asymptotically almost automorphic mild solution.

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